Harmonic Analysis of Exponential Nonlinear Devices and Validity of THD in Judging Nonlinearities

Nay Oo, Francois X. Nsabimana, Thomas Rohdenburg, and Woon-Seng Gan

***Abstract*—Derivations of generalized closed-form harmonic equations for a family of polynomial-approximated and parameterized exponential nonlinear devices (NLDs) are presented. The application of this research is for nonlinear signal processing based psychoacoustic bass enhancement systems. The derived harmonic equations are used to compute THD scores analytically to show that even though the nonlinear curves are different, which may cause different perceptual effects, the THD scores turn out to be exactly the same for all six exponential NLDs. The insights gained from this mathematical analysis indicate that, even without linking to perceptual attributes such as audio quality or nonlinear distortion perception, THD is not a suitable metric to judge or measure the quantitative degrees of nonlinear curves.**

***Index Terms*—Total harmonic distortion, nonlinear systems, nonlinear devices**

I. Introduction

Exponential NLDs are used in the signal processing systems for bass enhancement using psychoacoustic technique [1], [2]. For example, the NLD used for bass enhancement in multiactuator panels, a special type of flat panel loudspeakers to reproduce spatial audio under the wave field synthesis system is an exponential equation:

$$f(x) = e^{\alpha(x-1)} \tag{1}$$

where x is the input, f is the nonlinear function, and α is a gain factor that controls the harmonics [1]. However, quantitative or algebraic closed-form analysis of how α can be used to control harmonics is not available in the paper. Motivated by the curiosity of how α controls the harmonics, the mathematical analysis is started in this paper. Based on this analytical result, a more general analysis of base-parameterized exponential NLDs is carried out. Furthermore, min-max normalization technique (that is usually used to normalize scores in statistical analysis of data in the field of data mining) is applied here to transform the curves of exponential NLDs (i.e., the same formula is applied here for different usage). After the closed-form equations are obtained, whether total harmonic distortion (THD) metric can be used to judge the degree of nonlinearities is assessed here. The astounding result indicates that all these exponential NLDs deliver the same THD scores. Therefore, the validity of THD for the case of nonlinear device assessment is questioned.

Manuscript received May 3, 2012; revised June 4, 2012.

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II. Derivation of Harmonic Equations for Exponential Nonlinear Devices

First, the harmonic analysis equation is derived for (1) as follows. Assuming that all the exponential nonlinearities can be polynomial approximated, let us now define

$$\hat{f}\left[A\cos(\omega t)\right] = \frac{\hat{c}_0}{2} + \sum_{k=1}^{Q} \hat{c}_k \cos(k\omega t) \tag{2}$$

where $\hat{f}(x)$ is the polynomial approximated function of $f(x)$, with $x = A\cos(\omega t)$, where A is the amplitude and ω is the angular frequency of the input. The right hand side of (2) is a truncated Fourier series, where $\hat{c}_0/2$ is DC, $\hat{c}_k$ (for $k = 1, 2, \ldots, Q$) are the amplitudes of harmonics, where k is the harmonic index, Q is the highest order harmonic number, which is equivalent to the highest degree of approximating polynomials. The goal is to derive closed-form algebraic equations for $\hat{c}_k$ with $k = 0, 1, 2, \ldots, Q$ given the parameter of system equation and input signal amplitude.

From (1), $f(x) = \beta e^{\alpha x}$, is derived, where $\beta = 1/e^{\alpha}$ is a constant, parameterized by α. Using Taylor's series, let us denote

$$f_{\text{EXP0}}(x,\alpha) \triangleq \beta e^{\alpha x} \approx \hat{f}_{\text{EXP0}}(x,\alpha) \triangleq \beta \sum_{k=0}^{Q} \frac{\alpha^k}{k!} x^k \tag{3}$$

where EXP0 is the name given to (1). Based on the works of Schaefer and Suen in [3] and [4] that are, however, not in closed-forms due to the series expansions instead of polynomials, the closed-form harmonic equation for EXP0 is obtained as:

$$\hat{c}_k\big|_{\text{EXP0}} = 2\beta\left(\alpha A/2\right)^k \sum_{j=0}^{\lfloor (Q-k)/2 \rfloor} \frac{\left(\alpha A/2\right)^{2j}}{j!\,\Gamma(k+j+1)} \tag{4}$$

where $\Gamma(k+j+1) = (k+j)!$ is a gamma function and $\lfloor\ \rfloor$ is a floor function. Hence, (4) is the resultant harmonic analysis equation for (1).

Next, instead of using the parameter α, the base of the exponential function b is set as a parameter to control the harmonics, and the respective closed-form harmonic equations are derived. The two exponential NLDs $f(x) = b^x$ and $f(x) = b^{-x}$ are studied first. As in (3), let us define

$$f_{\mathrm{EXP1}}(x,b) \triangleq b^{x} = e^{x\ln b} \approx \hat{f}_{\mathrm{EXP1}}(x,b) \triangleq \sum_{k=0}^{Q} \frac{(\ln b)^{k}}{k!} x^{k} \quad (5)$$

$$f_{\mathrm{EXP2}}(x,b) \triangleq b^{-x} = e^{-x\ln b} \approx \hat{f}_{\mathrm{EXP2}}(x,b) \triangleq \sum_{k=0}^{Q} \frac{(-\ln b)^{k}}{k!} x^{k} \quad (6)$$

where EXP1 and EXP2 are names given to $f(x) = b^{x}$ and $f(x) = b^{-x}$ respectively. Following the analytical steps from (3) to (4), the harmonic equations for (5) and (6) are respectively obtained as follows:

$$\hat{c}_{k}\big|_{\mathrm{EXP1}} = 2\left(A\ln b/2\right)^{k} \sum_{j=0}^{\lfloor (Q-k)/2 \rfloor} \frac{\left(A\ln b/2\right)^{2j}}{j!\,\Gamma(k+j+1)} \quad (7)$$

$$\hat{c}_{k}\big|_{\mathrm{EXP2}} = 2\left(-A\ln b/2\right)^{k} \sum_{j=0}^{\lfloor (Q-k)/2 \rfloor} \frac{\left(A\ln b/2\right)^{2j}}{j!\,\Gamma(k+j+1)} \quad (8)$$

The input-output (IO) plots of (5) and (6) are shown in Fig. 1 where the parameter b is varying from 1.1 to 10 with a step-size of 1.1. Notice also from Fig. 1 that these NLDs are not suitable to be used directly in digital audio processing system, unless the output dynamic range is limited from -1 to 1 [i.e., to 0 dB full scale (dBFS)], to avoid clipping. Next section solves this problem by normalizing the nonlinear curves but the derived equations (7) and (8) are found to be reused.

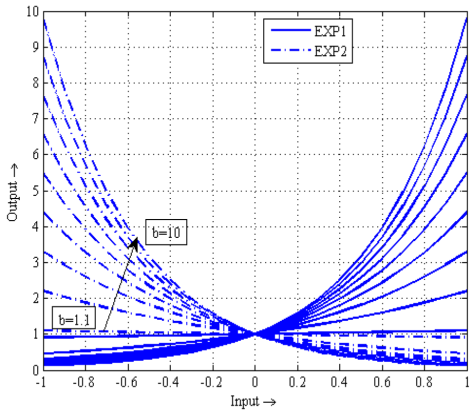

Fig. 1. Nonlinear transfer characteristics of EXP1 (5) and EXP2 (6). The parameter b is varied from 1.1 to 10 with stepsize of 1.1.

III. Min-Max Normalization for Exponential Nonlinear Curves

Min-max normalization is a statistical data preparation technique used in biometrics [6] or data mining [7]. The data transformation is linear on the original data [7]. The original formula for the min-max normalization is given by

$$\tilde{y} = \frac{y - y_{\min}}{y_{\max} - y_{\min}} (\tilde{y}_{\max} - \tilde{y}_{\min}) + \tilde{y}_{\min} \quad (9)$$

where min-max normalization maps a value y in the range [$y_{\min}$, $y_{\max}$] to $\tilde{y}$ in the range [$\tilde{y}_{\min}$, $\tilde{y}_{\max}$]. Instead of mapping the data, nonlinear functions can also be normalized to map the output range from [$\min[f(x)]$, $\max[f(x)]$] to [$\min[\tilde{f}(x)]$, $\max[\tilde{f}(x)]$], where $\tilde{f}(x)$ is a resultant function and $f(x)$ is a function to be normalized. Thus, by applying (9) to (5) and (6), the four closed-form functions of new exponential NLDs are obtained, which are bounded in the full-range (FR) of [-1, 1] or the half-range (HR) of [0, 1]. The functions are listed in the second column of Table 1, where their respective acronyms are in the first column. These analytical formulae can be further simplified by introducing two basic functions such as

$$\Phi_{1}(b) = \frac{b}{b^{2}-1}; \quad \Phi_{2}(b) = \frac{-1}{b^{2}-1} \quad (10)$$

So that the third column of Table I is obtained. The IO plots of these NLDs are shown in Fig. 2. Unlike the IO plots in Fig. 1, notice that the output is always bounded in between -1 to 1 or 0 to 1 regardless of varying parameter b. Notice the similarities between the four IO plots and the output dynamic ranges.

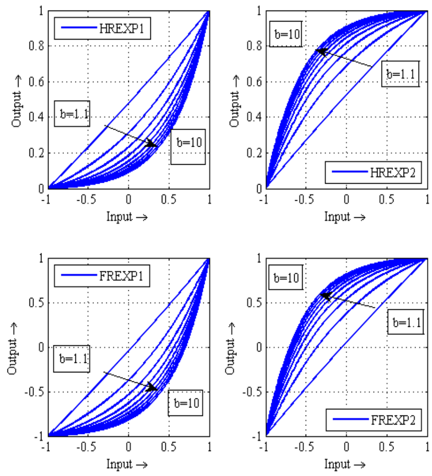

Fig. 2. Nonlinear transfer characteristics of HREXP1, HREXP2, FREXP1 and FREXP2. Parameter b varies from 1.1 to 10 with a stepsize of 1.1. Increase in parameter b increases the nonlinear curvatures.

TABLE I: Min-Max Normalized Exponential NLDs.

NLD (λ)	Nonlinear Function $f_{\lambda}(x,b)$	
HREXP1	$\frac{b^{x}-b^{-1}}{b-b^{-1}}$	$\Phi_{1}(b) f_{EXP1}(x,b) + \Phi_{2}(b)$
HREXP2	$\frac{b^{2}-b^{1-x}}{b^{2}-1}$	$b[\Phi_{2}(b) f_{EXP2}(x,b) + \Phi_{1}(b)]$
FREXP1	$2\left(\frac{b^{x}-b^{-1}}{b-b^{-1}}\right)-1$	$2 f_{HREXP1}(x,b) - 1$
FREXP2	$2\left(\frac{b^{2}-b^{1-x}}{b^{2}-1}\right)-1$	$2 f_{HREXP2}(x,b) - 1$

Harmonic analysis equations for these four NLDs of half-range and full-range are then derived and the results are listed in Table II. The second and third columns of Table II denote the truncated Fourier coefficients in (2). Notice also that the basis functions in (10) are embedded in the harmonic analysis equations and the simpler harmonic relationships among NLDs are revealed.

TABLE II: DERIVED HARMONIC EQUATIONS.

NLD (λ)	Truncated Fourier Coefficients	
	$\hat{c}_0\big\|_{\lambda}$	$\hat{c}_{k>0}\big\|_{\lambda}$
HREXP1	$2\left[\Phi_1(b)\dfrac{\hat{c}_0\big\|_{EXP1}}{2}+\Phi_2(b)\right]$	$\Phi_1(b)\hat{c}_{k>0}\big\|_{EXP1}$
HREXP2	$2\left\{b\left[\Phi_2(b)\dfrac{\hat{c}_0\big\|_{EXP2}}{2}+\Phi_1(b)\right]\right\}$	$b\Phi_2(b)\hat{c}_{k>0}\big\|_{EXP2}$
FREXP1	$2\left(\hat{c}_0\big\|_{HREXP1}-1\right)$	$2\hat{c}_{k>0}\big\|_{HREXP1}$
FREXP2	$2\left(\hat{c}_0\big\|_{HREXP2}-1\right)$	$2\hat{c}_{k>0}\big\|_{HREXP2}$

IV. VALIDITY OF THD METRIC IN JUDGING NONLINEARITIES

The THD metric [5] is defined as

$$\mathrm{THD}=\frac{\sqrt{\sum_{k=2}^{Q}\hat{c}_k^2}}{\left|\hat{c}_1\right|} \tag{11}$$

where $\hat{c}_k$ are truncated Fourier coefficients or amplitudes of harmonics. Note that $\hat{c}_k$ can be positive (same phase with the input) or negative (180 degree out of phase). In order to eliminate this phase information, the denominator of (11) is taken as the absolute value of the amplitude of the first harmonic (magnitude of the first harmonic), instead of just using the amplitude. However, root mean square (*rms*) values can also be used to compute THD metric.

The THD simulation results of the above six exponential NLDs such as $\gamma\in$ {EXP1, EXP2, HREXP1, HREXP2, FREXP1, FREXP2} are shown in three dimensional plots of Fig. 3, where the magnitude of input tone A and exponential base parameter b are varied from -60 to 0 dBFS with -1 dBFS incremental step-size, and 1.1 to 10 with a step-size of 1.1, respectively. Notice that all six plots look the same, meaning that computed numerical results of THD scores are exactly the same for all six NLDs. In order to explain why all these NLDs produce the same THD results even though their curves look different, an analysis is presented here. By using (7) and (8), the third column of Table II, and noticing that, $abs(c_k|_{\mathrm{EXP1}})=abs(c_k|_{\mathrm{EXP2}})$, where $abs(.)$ is an absolute operator, and $\Phi_1(b)=-b\Phi_2(b)$ from (10), THD analytical expression for the six exponential NLDs, parameterized by A and b is finally obtained as

$$\mathrm{THD}_{\gamma}(A,b)=\frac{\sqrt{\sum_{k=2}^{Q}\left(\hat{c}_k\big|_{\mathrm{EXP1}}\right)^2}}{abs\left(\hat{c}_1\big|_{\mathrm{EXP1}}\right)} \tag{12}$$

Therefore, it has been proven analytically that THD scores of all six exponential NLDs can be formulated as (12).

As for numerical values comparison, Table 3 provides the *rms* values of the five harmonics and their respective THD scores. Notice that THD scores are all equal to 35.6% for all six NLDs. The THD metric in (11) is used with individual harmonics' *rms* values, which are listed from second to sixth columns of Table 3. The exponential NLDs are polynomial approximated to the fifth order. The input signal peak amplitude is set as -1 dBFS and the base parameter is fixed as 5.55.

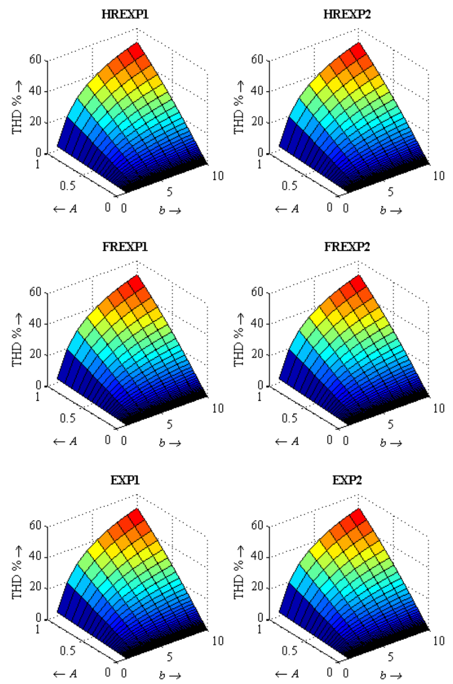

Fig. 3. THD plots of the six exponential NLDs.

TABLE III: NUMERICAL RESULTS OF THD SCORES USING RMS VALUES OF HARMONICS WITH $A=0.8913$, $b=5.5500$, AND $Q=5$ PARAMETER SETTINGS.

NLD (λ)	$c_k(\mathrm{rms})=\text{Peak amplitude}/\sqrt{2}=\|c_k\|/\sqrt{2}$					
	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	THD_{λ}
EXP1	1.4256	0.4926	0.1203	0.0200	0.0031	35.6 %
EXP2	1.4256	0.4926	0.1203	0.0200	0.0031	35.6 %
HREXP1	0.2655	0.0917	0.0224	0.0037	0.0006	35.6 %
HREXP2	0.2655	0.0917	0.0224	0.0037	0.0006	35.6 %
FREXP1	0.5310	0.1835	0.0448	0.0075	0.0011	35.6 %
FREXP2	0.5310	0.1835	0.0448	0.0075	0.0011	35.6 %

V. CONCLUSIONS

Based on the obtained results in Section IV, the conclusion is drawn that THD may not be a figure of merit to measure the different nonlinearities' transfer function characteristics because in our case all of the six exponential NLDs generate the same THD scores although their curvatures are different. In addition to this finding, closed-form algebraic harmonic analysis formulae for the exponential NLDs have been derived and presented in this paper. The min-max normalization from the data mining is applied to the transformation of nonlinear curves as an intermediate step for the analysis or derivations of new exponential NLDs that are bounded in HR or FR. Although THD results prove to be invalid to judge different nonlinearities here, the derived harmonic equations are novel to accurately compute the truncated Fourier coefficients or simply harmonic amplitudes and DC components of these polynomial approximated exponential NLDs for the application of psychoacoustic bass enhancement systems. The main objective of this paper is on the analysis of THD metric. For the further research to relate to perceptual domain, we would perform the comparative studies applying two perceptual models such as *Rnonlin* [8] and *PEMO-Q* [9], and multitone distortion analysis [10].

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