On Harmonic Addition Theorem

Nay Oo and Woon-Seng Gan

***Abstract*—When any arbitrary number of sinusoids with same frequency but different amplitudes and phases are added together, the resultant sinusoid has the same frequency. We derive a closed form expression and provide two different approaches to prove the theorem.**

***Index Terms*—Harmonic addition theorem, phasor addition theorem, phasor addition rule**

I. INTRODUCTION

Addition of sinusoids with the same frequency but arbitrary amplitudes and phases is the fundamental but important operation in acoustics, music, communication, and audio signal processing applications. Let us define

$$\begin{cases} x_s(t) = \sum_{i=1}^{L} \alpha_i \sin(\omega_0 t + \varphi_i), \\ x_c(t) = \sum_{i=1}^{L} \alpha_i \cos(\omega_0 t + \varphi_i), \end{cases} \tag{1}$$

where L denotes the total number of tonal components that are to be added, ω_0 is the angular frequency which is common to all the tonal components, and α_i , φ_i are amplitudes and phases, respectively. Notice that ω_0 is the same for all the sinuoids. Let the sinusoid and cosinusoid be respectively denoted as $x_s(t)$ and $x_c(t)$. Thus (1) can be written as

$$\begin{cases} x_s(t) = \beta \sin(\omega_0 t + \psi), \\ x_c(t) = \beta \cos(\omega_0 t + \psi), \end{cases} \tag{2}$$

where β is the amplitude and ψ is the phase. The mathematical problem of interest here is that given(1), β and ψ are to be formulated in closed-form formulae, parameterized by L, α_i , and φ_i for $i = 1, 2, \ldots, L$.

In signal processing literature, (2) is obtained from (1) using *Phasor Addition Rule* or *Phasor Addition Theorem* [1]. This is in fact equivalent to *Harmonic Addition Theorem* [2] in mathematics and here we prove the theorem from two different approaches and show this link. From that, we observe some interesting patterns in symbolic expansions; by using them, we formularize the closed-form expressions for β and ψ .

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II. HARMONIC ADDITION THEOREM

The Phasor Addition Theorem or Rule is equivalent to Harmonic Addition Theorem, and the formulae in the theorem are derived in this paper from different perspectives.

Theorem: Given the signal function $x_s(t) = \sum_{i=1}^{L} \alpha_i \sin(\omega_0 t + \varphi_i)$ or $x_c(t) = \sum_{i=1}^{L} \alpha_i \cos(\omega_0 t + \varphi_i)$, it is possible to find β and ψ so that $x_s(t) = \beta \sin(\omega_0 t + \psi)$ or $x_c(t) = \beta \cos(\omega_0 t + \psi)$ where

$$\begin{cases} \beta = \sqrt{\sum_{i=1}^{L} \alpha_i^2 + 2\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} \alpha_i \alpha_j \cos(\varphi_i - \varphi_j)}, \\ \psi = \arg\left(\frac{\sum_{i=1}^{L} \alpha_i \sin\varphi_i}{\sum_{i=1}^{2} \alpha_i \cos\varphi_i} \right), -\pi < \psi \le \pi. \end{cases} \tag{3}$$

A. Sub-Problem

To prove the theorem and to formulate (3), we define a sub-problem to obtain a formula for the addition of complex numbers in polar form without being necessary to convert them into rectangular form at all. This sub-problem is defined as follows. Let $x_e(t)$ be denoted as a complex exponential signal function that is given by

$$x_e(t) = \sum_{i=1}^{L} \alpha_i e^{j\varphi_i} = \sum_{i=1}^{L} \alpha_i \angle \varphi_i \tag{4}$$

where α_i is the amplitude and φ_i is the phase of the i^{th} complex number in polar form or in phasor notation. L is the total number of complex numbers to add, the objective is to convert (4) into

$$x_e(t) = \beta e^{j\psi} = \beta \angle \psi \tag{5}$$

where β is the amplitude and ψ is the phase. In other words, the problem is to formulate β and ψ of (5) in terms of α_i and φ_i for $i = 1, 2, \ldots, L$ of (4).

To solve this sub-problem, starting from $L = 2$, for $x_e(t) = \sum_{i=1}^{L} \alpha_i \angle \varphi_i = \beta \angle \psi$,

For the computation of ψ , *atan2* function [3] can be used to exactly locate the angle in any of the four quadrants in the complex plane. Notice that the ordinary *atan* function [3] range is, however, $-\pi/2 < \psi \le \pi/2$ in contrast to the *atan2* function range of $-\pi < \psi \le \pi$. To formulate the formulae of β and ψ , as stated in the sub-problem statement, we expand γ_L in (6) for $L = 2, 3, 4,$ as shown in Table I, and observe the following symbolic expansion pattern

$$\beta = \sqrt{(\alpha_1\cos\varphi_1 + \alpha_2\cos\varphi_2)^2 + (\alpha_1\sin\varphi_1 + \alpha_2\sin\varphi_2)^2} = \sqrt{\gamma_2} = \sqrt{\alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2\cos(\varphi_1 - \varphi_2)},$$
$$\psi = \operatorname{atan2}\left(\sum\nolimits_{i=1}^{2}\alpha_i\sin\varphi_i, \sum\nolimits_{i=1}^{2}\alpha_i\cos\varphi_i\right), \text{ for } -\pi < \psi \le \pi. \tag{6}$$

TABLE I: ALGEBRAIC AND TRIGONOMETRIC EXPANSIONS OF γ_L FOR $L = 2,3,4$

L	γ_L
2	$= \overbrace{\alpha_1^2 + \alpha_2^2}^{\xi_2} + 2\overbrace{\alpha_1\alpha_2\cos(\varphi_1 - \varphi_2)}^{\lambda_2}$
3	$= \overbrace{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}^{\xi_3} + 2[$ $\left.\begin{array}{l}\alpha_1\alpha_2\cos(\varphi_1-\varphi_2) + \alpha_1\alpha_3\cos(\varphi_1-\varphi_3) + \\ \alpha_2\alpha_3\cos(\varphi_2-\varphi_3)]\end{array}\right\}\lambda_3$
4	$= \overbrace{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2}^{\xi_4} + 2[$ $\left.\begin{array}{l}\alpha_1\alpha_2\cos(\varphi_1-\varphi_2) + \alpha_1\alpha_3\cos(\varphi_1-\varphi_3) + \alpha_1\alpha_4\cos(\varphi_1-\varphi_4) + \\ \alpha_2\alpha_3\cos(\varphi_2-\varphi_3) + \alpha_2\alpha_4\cos(\varphi_2-\varphi_4) + \\ \alpha_3\alpha_4\cos(\varphi_3-\varphi_4)]\end{array}\right\}\lambda_4$

From Table I, we define

$$\gamma_L = \left(\sum_{i=1}^{L}\alpha_i\cos\varphi_i\right)^2 + \left(\sum_{i=1}^{L}\alpha_i\sin\varphi_i\right)^2 = \xi_L + 2\lambda_L, \tag{7}$$

where

$$\xi_L = \sum_{i=1}^{L}\alpha_i^2, \tag{8}$$

$$\lambda_L = \sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\alpha_i\alpha_j\cos(\varphi_i - \varphi_j). \tag{9}$$

Thus, from (7), (8), and (9), β and ψ in (5) can now be formulated as

$$\beta = \sqrt{\gamma_L} = \sqrt{\sum_{i=1}^{L}\alpha_i^2 + 2\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\alpha_i\alpha_j\cos(\varphi_i - \varphi_j)},$$
$$\psi = \operatorname{atan2}\left(\sum\nolimits_{i=1}^{L}\alpha_i\sin\varphi_i, \sum\nolimits_{i=1}^{L}\alpha_i\cos\varphi_i\right), -\pi < \psi \le \pi. \tag{10}$$

Indices, $\{i, j\}$, of (9) can also be further observed as a *triangular number expansion pattern* as shown in Table 2. The total number of components and the total number of summations in (9) can be formulated as $T(n)$ and $T(n)-1$, respectively where $T(n)$ is the *triangular number* [4] and $n = L-1$, i.e.,

$$T(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \binom{n+1}{2} = {}_LC_2. \tag{11}$$

For the both cases of positive and negative φ_i's in (5), we define

$$\begin{cases} x_{e^-}(t) = \sum_{i=1}^{L}\alpha_i e^{-j\varphi_i} = \sum_{i=1}^{L}\alpha_i\angle -\varphi_i = \beta e^{-j\psi}, \\ x_{e^+}(t) = \sum_{i=1}^{L}\alpha_i e^{j\varphi_i} = \sum_{i=1}^{L}\alpha_i\angle\varphi_i = \beta e^{j\psi}. \end{cases} \tag{12}$$

with β and ψ as formulated in (6).

TABLE II: INDICES $\{i, j\}$ OF (9) FOR $L = 2,3,4$, AND $m \ge 2$

L	$\{i, j\}$	${}_LC_2$
2	$\{1,2\}$	1
3	$\{1,2\}, \{1,3\}$	3
	$\{2,3\}$	
4	$\{1,2\}, \{1,3\}, \{1,4\}$	6
	$\{2,3\}, \{2,4\}$	
	$\{3,4\}$	
$\vdots$	$\vdots$	$\vdots$
m	$\{1,1+1\}, \{1,1+2\}, \ldots, \{1,m-1\}, \{1,m\}$	${}_mC_2$
	$\{2,2+1\}, \{2,2+2\}, \ldots, \{2,m\}$	
	$\{3,3+1\}, \ldots, \{3,m\}$	
	$\vdots$	
	$\{m-2,m-1\}, \{m-2,m\}$	
	$\{m-1,m\}$	

B. Proof I

In this sub-section, the first proof of the harmonic addition theorem is presented. Using (12) and Euler's formula, $x_s(t)=\sum_{i=1}^{L}\alpha_i\sin(\omega_0 t+\varphi_i)$

$$=\frac{1}{2j}\left[e^{j\omega_0 t}\underbrace{\sum_{i=1}^{L}\alpha_i e^{j\varphi_i}}_{x_{e^+}(t)}-e^{-j\omega_0 t}\underbrace{\sum_{i=1}^{L}\alpha_i e^{-j\varphi_i}}_{x_{e^-}(t)}\right]=\frac{1}{2j}\left[e^{j\omega_0 t}\underbrace{\beta e^{j\psi}}_{x_{e^+}(t)}-e^{-j\omega_0 t}\underbrace{\beta e^{-j\psi}}_{x_{e^-}(t)}\right]=\beta\sin(\omega_0 t+\psi),$$

and $x_c(t)=\sum_{i=1}^{L}\alpha_i\cos(\omega_0 t+\varphi_i)$

$$=\frac{1}{2}\left[e^{j\omega_0 t}\underbrace{\sum_{i=1}^{L}\alpha_i e^{j\varphi_i}}_{x_{e^+}(t)}+e^{-j\omega_0 t}\underbrace{\sum_{i=1}^{L}\alpha_i e^{-j\varphi_i}}_{x_{e^-}(t)}\right]=\frac{1}{2}\left[e^{j\omega_0 t}\underbrace{\beta e^{j\psi}}_{x_{e^+}(t)}+e^{-j\omega_0 t}\underbrace{\beta e^{-j\psi}}_{x_{e^-}(t)}\right]=\beta\cos(\omega_0 t+\psi).$$

C. Proof II

In this sub-section, the second proof of the harmonic addition theorem is presented. Using (12), and Fourier transform denoted by the notation $\overset{\mathbb{F}}{\Leftrightarrow}$,

$$x_s(t)=\sum_{i=1}^{L}\alpha_i\sin(\omega_0 t+\varphi_i)\overset{\mathbb{F}}{\Leftrightarrow}X_s(\omega)=\pi\delta(\omega-\omega_0)e^{-j\pi/2}\underbrace{\sum_{i=1}^{L}\alpha_i e^{j\varphi_i}}_{x_{e^+}(t)}+\pi\delta(\omega+\omega_0)e^{j\pi/2}\underbrace{\sum_{i=1}^{L}\alpha_i e^{-j\varphi_i}}_{x_{e^-}(t)}$$

$$=\pi\delta(\omega-\omega_0)e^{-j\pi/2}\underbrace{\beta e^{j\psi}}_{x_{e^+}(t)}+\pi\delta(\omega+\omega_0)e^{j\pi/2}\underbrace{\beta e^{-j\psi}}_{x_{e^-}(t)}$$

$$=\beta\left[\pi\delta(\omega-\omega_0)e^{j(-\pi/2+\psi)}+\pi\delta(\omega+\omega_0)e^{j(\pi/2-\psi)}\right]$$

$$\overset{\mathbb{F}}{\Leftrightarrow}x_s(t)=\beta\sin(\omega_0 t+\psi).$$

Similarly,

$$x_c(t)=\sum_{i=1}^{L}\alpha_i\cos(\omega_0 t+\varphi_i)\overset{\mathbb{F}}{\Leftrightarrow}X_c(\omega)=\pi\delta(\omega-\omega_0)\underbrace{\sum_{i=1}^{L}\alpha_i e^{j\varphi_i}}_{x_{e^+}(t)}+\pi\delta(\omega+\omega_0)\underbrace{\sum_{i=1}^{L}\alpha_i e^{-j\varphi_i}}_{x_{e^-}(t)}$$

$$=\pi\delta(\omega-\omega_0)\underbrace{\beta e^{j\psi}}_{x_{e^+}(t)}+\pi\delta(\omega+\omega_0)\underbrace{\beta e^{-j\psi}}_{x_{e^-}(t)}$$

$$=\beta\left[\pi\delta(\omega-\omega_0)e^{j\psi}+\pi\delta(\omega+\omega_0)e^{-j\psi}\right]$$

$$\overset{\mathbb{F}}{\Leftrightarrow}x_c(t)=\beta\cos(\omega_0 t+\psi).$$

In the above proof, $\delta(.)$ is the Dirac delta function. Hence, the harmonic addition theorem has been proved.

III. Conclusions

In conclusion, the harmonic addition theorem is proved from two different approaches, and closed-form formula to calculate the exact amplitude, and phase of the resultant tone are derived from the first principles. A pattern of symbolic expansions is observed, and noted the relation to the trigonometric number, which in turn shows the computational load by predicting the number of summations needed to perform as derived in (11). Based on this work, the resultant formulae described in (3) can be easily converted into an algorithm to compute the exact amplitudes, and phases of the resultant tone (sinusoids) when many tones with the same frequency but arbitrary amplitudes, and phases are linearly added together. Moreover, the mathematical results in [2] do not include the closed-form formulation of the harmonic addition theorem. In this paper, the exact closed-form formula in (3) (see (27) and (28) in [2]) is obtained with two separate proofs. In fact, this closed-formed formula has been applied to compute the exact harmonic output [5] and intermodulation distortion output [6] (in terms of amplitude, frequencies and phase) from polynomial-approximated static nonlinearities in the area of audio engineering.

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